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among my autographs. There is nothing to show to whom it was written, but it refers to one phase of the famous Libri controversy which aroused great interest and very bitter feeling at the beginning of the second half of the nineteenth century. Libri, the well-known historian of mathematics, had been accused of stealing many of the books that made up his extensive library. The French courts had, in his absence, convicted him of the crime. He had fled to London and had become an intimate friend of the De Morgans, and this letter shows the faith that De Morgan himself had in the innocence of the accused. It also shows the esteem in which De Morgan held Chasles, the French geometer, and the lengths to which a question of principle could carry him in what was, as regards the scientists of the time, an international dispute.

The letter, which has not before been published to my knowledge, is as follows:

7 Camden Street
March 18/54

My dear Sir:

I am fully of opinion that M. Chasles has high claims on the Royal Society—But, though I am aware it is useless, I for one, will never give an opinion on his claims except in connexion with those of another, whose claims I hold still higher—I mean *Libri*. What I should say of Chasles as an historian, I should say of Libri in a higher degree, and both are truly original mathematicians. I am well aware that though the malignant accusations made against Libri have been refuted in a way in which charges very seldom are refuted—and though the evidence of determination to prevent his having a fair trial is as clear as any evidence can be—yet the power of ignoring notorious facts which bodies of men possess in their collective capacity,—and the disposition of our country to acknowledge all results of foreign law, however little it may agree with our own in fairness of procedure—would make it useless to propose Libri to the R.S. I myself should not entertain a sufficiently high opinion of any English Society to venture on such a step.

This being understood, and speaklg [*sic*, apparently for "speaking"] of Chasles absolutely—I should describe him as one of the very few mathem^{ns}—and the only one of French birth—who attend to the history of the science. In this subject he is a man of real learning—deep in original sources. His *Apercu* &c is but one of his contributions—and all of them throw new light on the older history of geometry & arithmetic.

In mathematics—as you know—he is among the foremost of those who have developed the powers of geometry to that extent which would have left algebra behind as an instrument—if algebra itself had not received new developments in the mode of applying it to geometry.

Chasles, I have no doubt, has been underrated by his countrymen. In truth, who was there in France (except Libri) to appreciate him? But if ever history and geometry revive in that country—Chasles will be looked upon as the founder of a school.

Chasles should have been elected years ago—He is now in the Institute—You might have given his passport—you can now only *viser* it—This is the great fault of our medals, associateships &c—and springs partly from moral fear, partly from it being nobody's business to form an independ^{t} opinion on reputations.

Yours very truly,
A. De Morgan

QUESTIONS AND DISCUSSIONS.

Edited by C. F. Gummer, Queen's University, Kingston, Ont., Canada.

REPLIES.

Of the two replies to Question 44, that of Professor Gilman contains a solution of the problem proposed, in the shape of a simple formula for the co-factors in the given "binomial" determinant.

Miss Hazlett does not consider the simplification of the co-factors in the particular case, but establishes a certain general formula for the co-factors in a determinant whose elements are all zero to the right of the principal diagonal. This is essentially the same as evaluating a determinant in which all elements to the right of the principal diagonal are zero with the exception that those in the partial diagonal next to the principal diagonal are all unity. Particular determinants of this class are continuants[1] and the expressions for the sums of like powers in terms of elementary symmetric functions.[2]

44. (*1921*, 260). Is there any known formula for the co-factor of the element a_{ij} in the "binomial" determinant here shown?

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ C_1{}^2 & 1 & 0 & \cdots & 0 & 0 \\ C_2{}^4 & C_1{}^4 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n{}^{2n} & C_{n-1}{}^{2n} & C_{n-2}{}^{2n} & \cdots & C_1{}^{2n} & 1 \end{vmatrix}.$$

Calling the co-factors A_{ij}, it is plain that $A_{ii} = 1$, and $A_{ij} = 0$, $(i > j)$. The co-factors of the zero elements, $(i < j)$, are undoubtedly expressible in a formula; the writer would like to know if any reader of the MONTHLY has met with such a formula anywhere.

I. Reply by R. E. Gilman, Brown University.

Solving for $\cos (2j - 2)\theta$ the system of equations[3]

$$\begin{aligned} \tfrac{1}{2} \cos^0 \theta &= \tfrac{1}{2} \\ 2 \cos^2 \theta &= 1 + \cos 2\theta \\ 2^3 \cos^4 \theta &= 3 + 4 \cos 2\theta + \cos 4\theta \\ \cdots & \cdots \\ 2^{2n-1} \cos^{2n} \theta &= \tfrac{1}{2}C_n{}^{2n} + C_{n-1}{}^{2n} \cos 2\theta + \cdots + \cos 2n\theta, \end{aligned}$$

we have

$$\cos (2j - 2)\theta = A_{1j}(\tfrac{1}{2} \cos^0 \theta) + A_{2j}(2 \cos^2 \theta) + \cdots.$$

Hence A_{ij} is the coefficient of $(2 \cos \theta)^{2i-2}$ in the expansion of $2 \cos (2j - 2)\theta$ in powers of $\cos \theta$, which may be written[4] for $j > i$

$$A_{ij} = (-1)^{j-i} \frac{(2j - 2)(j + i - 3)!}{(j - i)!(2i - 2)!}.$$

II. Reply by Olive C. Hazlett, Mount Holyoke College.

It is not difficult to derive a set of formulæ for the co-factors, A_{ij}, of any element, a_{ij}, of the determinant

$$a = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & a_{k3} & \cdots & 1 \end{vmatrix}.$$

As the correspondent remarks, $A_{ii} = 1$ and $A_{ij} = 0$ when $i > j$. For the cases when $i < j$, a formula can be proved by induction.

The proof of this last is based on two facts: (1) the co-factor of a_{ij} (when $j \leqq k$, the order of the determinant) is independent of k; (2) the product of the matrix a by A' (the transpose of the matrix, A, of the co-factors A_{ij}) is the unit matrix, I. Statement (2) is well known, and (1) follows from the fact that all elements to the right of the main diagonal are zero. Accordingly, we first find a formula for A_{12} for the determinant of general order k by finding A_{12} when the order of the determinant is 2. Then we consider the determinant, a, of order 3 and use the formula for A_{12} just found. The necessary and sufficient conditions that $aA' = I$ give us the formulæ for A_{13} and A_{23}. Next we consider the determinant a of order 4 and thus derive formulæ for A_{14}, A_{24} and A_{34}. Finally, we proceed as in a proof by induction.

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[1] L. G. Weld, *The Theory of Determinants*, New York, 1893, p. 178.

[2] L. E. Dickson, *Elementary Theory of Equations*, New York, 1914, p. 169.

[3] Loney, *Plane Trigonometry*, Cambridge, 1893, Part II, p. 342.

[4] *Ibid.*, p. 349.

From the determinant a of order 2, we see that $A_{12} = -a_{21}$. Substituting this in the determinant a of order 3 (as we have a right to do, by statement (1)), we have

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & 1 & 0 \\ A_{13} & A_{23} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But this is true if and only if

$$\begin{cases} 0 = a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13}, \\ 0 = a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23}. \end{cases}$$

Hence

$$\begin{cases} A_{13} = -a_{31} + a_{32}a_{21}, \\ A_{23} = -a_{32}. \end{cases}$$

Substituting these formulæ for A_{12}, A_{13} and A_{23} in A', formed for the determinant a of order 4, we find that the necessary and sufficient conditions that $aA' = I$ are

$$0 = a_{41}A_{i1} + a_{42}A_{i2} + a_{43}A_{i3} + a_{44}A_{i4} \quad (i = 1, 2, 3);$$

and hence

$$\begin{cases} A_{14} = -a_{41} + a_{42}a_{21} + a_{43}a_{31} - a_{43}a_{32}a_{21}, \\ A_{24} = -a_{42} + a_{43}a_{32}, \\ A_{34} = -a_{43}. \end{cases}$$

Now proceed by induction. Assume that, for $f \leqq l$, it is true that whenever $g < f$,

$$A_{gf} = -a_{fg} + \Sigma a_{fi}a_{ig} - \Sigma a_{fi_1}a_{i_1i_2}a_{i_2g} + \cdots + (-1)^{f-g}a_{fi_1}a_{i_1i_2}a_{i_2i_3} \cdots a_{g+1,g},$$

where the summation signs are to be interpreted in a special way. The first summation, $\Sigma a_{fi}a_{ig}$, denotes the sum of all terms of the type indicated for which $f > i > g$. In the next summation, $f > i_1 > i_2 > g$. The third summation, which is not written above, is $\Sigma a_{fi_1}a_{i_1i_2}a_{i_2i_3}a_{i_3g}$ and means the sum of all terms of the type indicated for which $f > i_1 > i_2 > i_3 > g$. Similarly with all the summation signs. In each succeeding summation there is an additional factor in each term. If there is an odd number of factors, a_{st}, in a term, the sign to be used is "minus"; otherwise, "plus." In the last summation, there is necessarily only one term and in this case $i_1 = f - 1$, $i_2 = f - 2$, and more generally $i_r = f - r$.

Now let the order of the determinant a be $l + 1$. Since the elements of a and A' outside of the last row and column are the same as in the case of the determinant of order l and in both the elements of the last column are 0, 0, $\cdots$, 1, it follows from our assumption that the product aA' is just like the unit matrix of order $l + 1$ in its first l rows, and to make the last row the same it is only necessary that the l equations

$$0 = \sum_{f=1}^{l+1} a_{l+1,f}A_{gf} \quad (g \neq l + 1)$$

be satisfied. Substituting in these equations the expressions for the A_{gf} in which $f \leqq l$, we derive formulæ for the $A_{g,l+1}$.

It is not hard to see that the formula for $A_{g,l+1}$ is of the same form as those for A_{gf} $(f \leqq l)$. First, every term is a product of a number of a_{ij}'s of the general form

$$a_{l+1,i_1}a_{i_1i_2} \cdots a_{i_{s-1}i_s}a_{i_sg}, \tag{I}$$

in which $l + 1 > i_1 > i_2 > \cdots > i_s > g$. This follows from the fact that every term comes from a product $a_{l+1,f}A_{gf}$ and from the fact that every term of A_{gf} $(f \leqq l)$ is a product of a's of the form

$$a_{fj_1}a_{j_1j_2} \cdots a_{j_{t-1}j_t}a_{j_tg}, \tag{II$_f$}$$

in which $f > j_1 > j_2 > \cdots > j_t > g$. Moreover, we get every term of the type I in which $s = 1, 2, \cdots, l - g$, since each A_{gf} contains every term of the type II$_f$ for which $t = 1, \cdots, l - g - 1$. Also, we get no term twice. Finally, the signs are as they should be, for in the expression for $A_{g,l+1}$ each A_{gf} $(f \leqq l)$ is multiplied by an $a_{l+1,f}$ and the sign of the product is changed. Thus the induction is complete.

III. Note by the EDITOR.

Miss Hazlett's formula for the co-factor of a_{ij} $(i < j)$ in the determinant a is also readily obtained as follows:

The system of equations

$$\begin{array}{ll} x_1 & = y_1 \\ a_{21}x_1 + x_2 & = y_2 \\ a_{31}x_1 + a_{32}x_2 + x_3 & = y_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + x_k & = y_k \end{array}$$

has the unique solution $x_j = \Sigma_i A_{ij} y_i$, $j = 1, \cdots, k$.

But we also find by solving the equations in succession $x_1 = y_1$, $x_2 = y_2 - a_{21}y_1$, $x_3 = y_3 - a_{31}y_1 + a_{32}a_{21}y_1 - a_{32}y_2$, etc. Hence $A_{12} = - a_{21}$, $A_{13} = - a_{31} + a_{32}a_{21}$, $A_{23} = - a_{32}$, etc., which agree with Miss Hazlett's formula; and, if the formula is assumed to hold for $i < j < n$, it is easy, by solving the nth equation, to extend it to $j = n$.

DISCUSSIONS.

In the first of the following discussions Mr. Webb presents a method of deriving the addition formulæ of plane trigonometry in which the various functions involved appear as line-segments rather than as the ratios of sides of right triangles. The author maintains that certain advantages in the way of clearness and concreteness result from this treatment, based on the idea of "arc in unit circle," and not of "angle." It is hardly necessary to point out that the ancient preference for the arc as independent variable is by no means out of date, and is favored particularly by French writers. A well-known example is Serret's treatise,[1] in which the properties of the functions are developed in connection with the arc, and the definition of angle (as the measure of the arc in a unit circle) only introduced on page 92 in preparation for the work on the triangle. On the whole, modern teachers are disposed to lay more stress on the relation to similar figures, to encourage the sense of angle as difference in direction of lines (independent of the scale of measurement), and to regard the graduated circle, of any convenient radius, simply as a useful device for the measurement of angles at the center. This is perfectly natural, since it is seldom the arc of the circle in which we are really interested, and the idea of angle, whatever the geometric basis, has a certain imaginative quality not to be found in the arc of the one-inch circle. Thus the consideration that the short hand of the clock turns twice as quickly as the earth takes on a peculiarly unsuggestive aspect if we are required to measure the two angular speeds on the same circle, whether of three inches or of four thousand miles.

It must be remembered however that the difference between the view of an angle as arc in unit circle and the view of it as ratio of arc to radius disappears entirely when we consider that the unit adopted is perfectly arbitrary, and that in fact one of us is comparing the arc with the radius and the other is comparing it with a certain portion of the foot-rule used in constructing the figure.

It may be well to add that Mr. Webb's paper is not primarily an argument in favor of the "arc method," but is more particularly a method of presenting the addition formulæ.

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[1] J.-A. Serret, *Traité de Trigonométrie*, Paris, 1908.

In the second discussion Professor Reynolds outlines a method for teaching the mathematics of investment without the use of the geometric series. In view of the recommendation of the National Committee on Mathematical Requirements that this subject should be studied (by election) in the Secondary School, these suggestions will be of interest to others besides college teachers.

I. A METHOD OF DERIVING FORMULÆ FOR THE EXPANSION OF $\sin(x+y)$ AND $\cos(x+y)$.

BY H. E. WEBB, Central High School, Newark, N. J.

For some time past the writer, in beginning with his classes the subject of plane trigonometry, has defined the sine and the cosine of a *number* as lines in a unit circle (the number being measured in linear units on the circumference), while the tangent and cotangent are defined as the ratios of these to each other, and the cosecant and secant as their respective reciprocals. At a later stage these functions are related to ratios between the sides of a right triangle, when the number is less than $\pi/2$, and are regarded as functions of the central angle which intercepts the arc.[1] This procedure has numerous advantages, among which may be mentioned the presentation of a clear idea of the limiting cases, and of the nature of the discontinuity of the four functions last named; an early acquaintance with "radian measure"; an escape from the necessity of two sets of definitions, one for the acute angle and the other for other angles; and the opportunity for presenting clearly the notions of negative angles and of positive angles in excess of 180°.

In line with this general plan, the following development of the expansion of $\sin(x+y)$ and $\cos(x+y)$ is presented, with the apology that it has not, to the knowledge of the writer, previously appeared in printed form.

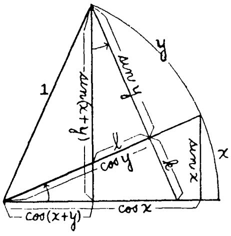

FIG. 1.

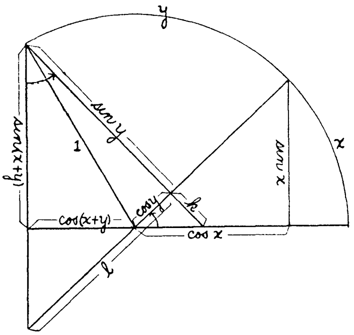

FIG. 2.

The radius of the circle is in each case unity. The notation of the figures is self-explanatory. From similar triangles, two proportions can be set up in each of the cases $\sin(x+y)$ and $\cos(x+y)$. For the case of $\sin(x+y)$,

$$k/\cos y = \sin x/\cos x, \tag{1}$$

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[1] This plan follows an outline of the subject by Professor C. O. Gunther, of Stevens Institute of Technology.